

# Mixed eigenvalues of $p$ -Laplacian on trees

Ling-Di WANG

School of Mathematics and Statistics, Henan University, Kaifeng 475001, China

**Abstract** The purpose of the paper is to present quantitative estimates for the principal eigenvalue of discrete  $p$ -Laplacian on the set of rooted trees. Alternatively, it is studying the optimal constant of a class of weighted Hardy inequality. Three kinds of variational formulas in different formulation for the mixed principal eigenvalue of  $p$ -Laplacian on the set of trees with unique root as Dirichlet boundary are presented. As their applications, we obtain a basic estimate of the eigenvalue on trees.

**Keywords**  $p$ -Laplacian eigenvalue, Dirichlet boundary, weighted Hardy inequality

**MSC** 60J60, 34L15

## 1 Introduction

In [3, 7], mixed principal eigenvalue for birth-death process on line were studied. Inspired by analogies research for that, mixed principal  $p$ -Laplacian on line were studied in [4, 5]. We shall extend the related results to a more general setting, investigating the quantitative estimates of the mixed principal  $p$ -Laplacian on trees with unique root as Dirichlet boundary. A basic result on the property of eigenvalue of  $p$ -Laplacian on trees, which is a key point for the extension, will be obtained.

By a tree, denote  $T$ , we mean  $T$  a undirected, connected, locally finite graph without cycles. One distinguished vertex, say  $o$ , is called the root. For any vertex  $i$ , the number of edges on the unique simple path between  $i$  and the root  $o$  is called the level of  $i$  and denote  $|i|$ . Let  $E$  be the edge set and  $V$  be the vertexes of  $T$ . The vertexes at level  $|i| + 1$  (correspondingly,  $|i| - 1$ ) that are adjacent to  $i$  are called children (correspondingly, parents) of  $i$ . Throughout the paper, we assume that trees are locally finite (i.e., each vertex has finite children).

To be specified,  $J(i)$  is the set of children of vertex  $i$  and  $i^*$  is a parent of  $i$ . Operator  $\Omega_p$  we focusing on in the paper is of the form

$$\Omega_p f(i) = \sum_{j \in J(i)} v_j |f_j - f_i|^{p-2} (f_j - f_i) + v_i |f_{i^*} - f_i|^{p-2} (f_{i^*} - f_i), \quad i \in V \setminus \{o\},$$

where  $\{v_i : i \in V\}$  is a positive sequence. We concentrate on estimating the  $p$ -Laplacian eigenvalue on a tree, which is described as follows:

$$\text{“Eigenequation”}: \quad \Omega_p g(k) = -\lambda \mu_k |g_k|^{p-2} g_k, \quad k \in V \setminus \{o\}; \quad (1)$$

$$\text{boundary conditions:} \quad g_o = 0. \quad (2)$$

where  $\{\mu_k : k \in V\}$  is a positive sequence and adopt the convention that  $\sum_{i \in \emptyset} f_i = 0$  for some sequence  $\{f_i\}$  throughout the rest of this paper. If  $(\lambda, g)$  with  $g \neq 0$  is a solution to the eigenvalue problem, then  $\lambda$  is called an  $p$ -Laplacian eigenvalue, and  $g$  is its eigenfunction. Especially, when  $p = 2$ , the eigenvalue corresponds to the decay rate of birth-death process on trees and  $\{\mu_k\}$  is just the invariant measure of birth-death process on trees (see [7]).

Define

$$D_p(f) = \sum_{i \in V \setminus \{o\}} v_i |f_i - f_{i^*}|^p, \quad f_o = 0.$$

Let  $(\lambda_p, g)$  be a solution to eigenequation (1) with boundary condition (2). It is well known that  $\lambda_p$  has the following classical variational formula

$$\lambda_p = \inf \{D_p(f) : \mu(|f|^p) = 1, f_o = 0\}, \quad (3)$$

We use the ordinary inner product

$$(f, g) = \sum_{k \in V} f_k g_k.$$

Then

$$D_p(g) = (-\Omega_p g, g).$$

Actually, for functions  $f$  and  $g$  with  $f_o = g_o = 0$ , we have

$$(-\Omega_p f, g) = - \sum_{i \in V} \sum_{j \in J(i)} v_j |f_j - f_i|^{p-2} (f_j - f_i) g_i - \sum_{i \in V} v_i |f_{i^*} - f_i|^{p-2} (f_{i^*} - f_i) g_i$$

By exchanging the order of sums, the formula equals to

$$- \sum_{j \in V \setminus \{o\}} \sum_{i=j^*} v_j |f_j - f_i|^{p-2} (f_j - f_i) g_i - \sum_{i \in V} v_i |f_{i^*} - f_i|^{p-2} (f_{i^*} - f_i) g_i.$$

By  $g_o = 0$ , we have

$$(-\Omega_p f, g) = \sum_{j \in V \setminus \{o\}} v_j |f_j - f_{j^*}|^{p-2} (f_{j^*} - f_j) (g_{j^*} - g_j).$$

Then the assertion holds by letting  $f = g$ .

Define

$$\mathcal{D}(D) = \{f : f \text{ is a real function defined on } V, f_o = 0, D_p(f) < \infty\}.$$

Formula (3) can be rewritten as the following weighted Hardy inequality:

$$\mu(|f|^p) \leqslant A D_p(f), \quad f \in \mathcal{D}(D),$$

with the optimal constant  $A = \lambda_p^{-1}$ . This explains the relationship between the p-Laplacian eigenvalues and the optimal constant of Hardy inequality.

For a tree  $T$ , denote by  $N$  ( $N \leqslant \infty$ ) the maximal level of tree  $T$  and  $T_i$  ( $i$  is included) is a subtree of tree  $T$  with  $i$  as root. Let

$$\Lambda_i = \{k \in V : |k| = i\}, \quad i \in \mathbb{Z}^+$$

be the set of elements in the  $i^{\text{th}}$  level of the tree. It is clear that  $\lambda_p > 0$  if  $N < \infty$  (otherwise,  $\Omega_p g(i) = 0$ ). By letting  $i \in \Lambda_N$  in (1), we have  $g_i = g_{i^*}$  for  $i \in \Lambda_N$ . By the induction, we have  $g_i = g_o = 0$  for  $i \in V$ , which is a contraction to  $g \neq 0$ ).

It is easy to see that  $\lambda_p = 0$  provided  $\sum_{k \in V} \mu_k = \infty$  by letting  $f_i = 1$  for  $i \in V \setminus \{o\}$  and  $f_o = 0$  in (3). Therefore, we always assume that  $\sum_{k \in V} \mu_k < \infty$ . Without loss of generality, we also assume that the root  $o$  has a child in the paper.

We mention that the methods used in this paper are mainly similar to that in [3], except one of the key proof of Lemma 2.1 below, in which the monotone of eigenfunction is proved for  $p \geqslant 2$ . Whether Lemma 2.1 still holds for  $p \in [1, 2)$  or not is still open for us which lead to that some equalities are uncertain in Theorem 2.2 below.

The paper is organized as follows. In Section 2, we present the main results, including the monotone of eigenfunction, three kinds of variational formulas for p-Laplacian eigenvalue and its applications (a quantitative estimates of the p-Laplacian eigenvalue). One example is presented at the end of Section 2. The sketch proofs of the main results are presented in Section 3.

## 2 Main results

To state our results, we need some notations. Let  $\mathcal{P}(i)$  be the set of all the vertexes (the root  $o$  is excluded) in the unique simple path from  $i \in V \setminus \{o\}$  to the root and  $V_i$  the set of vertexes of subtree  $T_i$  for some  $i \in V \setminus \{o\}$ . For  $p > 1$ , let  $\hat{p}$  be its conjugate number (i.e.,  $1/p + 1/\hat{p} = 1$ ). For  $i \in V \setminus \{o\}$ ,

define  $\hat{v}_j = v_j^{1-\hat{p}}$ , three operators which are parallel to those introduced in [5], as follows:

$$\begin{aligned} I_i(f) &= \frac{1}{v_i(f_i - f_{i^*})^{p-1}} \sum_{j \in V_i} \mu_j f_j^{p-1} \quad (\text{single summation form}), \\ II_i(f) &= \frac{1}{f_i^{p-1}} \left[ \sum_{k \in \mathcal{P}(i)} \hat{v}_k \left( \sum_{j \in V_k} \mu_j f_j^{p-1} \right)^{\hat{p}-1} \right]^{p-1} \quad (\text{double summation form}), \\ R_i(w) &= \mu_i^{-1} [v_i(1 - w_i^{-1})^{p-1} - \sum_{j \in J(i)} v_j(w_j - 1)^{p-1}] \quad (\text{difference form}). \end{aligned}$$

Similar operators were initially introduced in [1, 2, 3] respectively for birth-death process in dimension one. We adopt the convention that  $1/0 = \infty$  and  $1/\infty = 0$  throughout the paper. To study the lower estimates of p-Laplacian eigenfunction, based on the properties of eigenfunction presented in Lemma 2.1 below, the domains of the three operators are defined respectively as follows:

$$\begin{aligned} \mathcal{F}_I &= \{f : f_o = 0, f_i > f_{i^*} \text{ for } i \in V \setminus \{o\}\}, \\ \mathcal{F}_{II} &= \{f : f_o = 0, f > 0 \text{ on } V \setminus \{o\}\}, \\ \mathcal{W} &= \{w : w > 1, w_o = \infty\}. \end{aligned}$$

For the upper bounds, some modifications are needed to avoid non-summable problem, as shown below.

$$\begin{aligned} \widetilde{\mathcal{F}}_I &= \{f \geq 0 : f_o = 0, \exists k \in V \setminus \{o\} \text{ such that } f_i > f_{i^*} \text{ for } i \in \mathcal{P}(k), \text{ and } f_i = f_{i^*} \text{ for } |i| \geq |k|\}, \\ \widetilde{\mathcal{F}}_{II} &= \{f \geq 0 : f_o = 0, f \neq 0, \exists 1 \leq n < N+1 \text{ such that } f_i = f_{i^*} \text{ for } |i| \geq n+1\}, \\ \widetilde{\mathcal{W}} &= \bigcup_{m: 1 \leq m < N+1} \left\{ w : w_o = \infty, w_i > 1 \text{ and } \sum_{j \in J(i)} v_j(w_j - 1)^{p-1} < v_i(1 - w_i^{-1})^{p-1} \text{ for } |i| \leq m, \text{ and } w_i = 1 \text{ for } |i| \geq m+1 \right\}. \end{aligned}$$

In some extent, these functions are imitated of eigenfunctions of  $\lambda_p$ . To avoid the trivial estimates, we need a modified form of  $R$ , denote  $\widetilde{R}$ , acting on  $\widetilde{\mathcal{W}}$  by replacing  $\mu_i$  with  $\tilde{\mu}_i = \sum_{j \in V_i} \mu_j$  in  $R_i(w)$  when  $|i| = m$ , where  $m$  is the same one in  $\widetilde{\mathcal{W}}$ . Besides,  $\widetilde{R}$  is also used when operating the approximating procedure (at this time,  $\mu_i$  is replaced with  $\tilde{\mu}_i$  for each  $i \in V$ , see Step 4 in the proof of Theorem 2.2 below). Here and in what follows, the superscript “ $\sim$ ” means modified. The set below is also needed.

$$\widetilde{\mathcal{F}}_{II} = \{f \geq 0 : f_o = 0, f \neq 0, f II(f)^{\hat{p}-1} \in L^p(\mu)\}.$$

The following lemma presents us an important property of eigenfunction  $g$ , providing the basis for the choices of those test functions sets of operators  $I$ ,  $II$  and  $R$ . More details see the comments before Lemma 3.1 below.

**Lemma 2.1** *Let  $T$  be a tree (may have infinite vertexes) with vertexes set  $V$  and  $p \geq 2$ . If  $g \in L^p(\mu)$ ,  $g \neq 0$  and  $(\lambda_p, g)$  is a solution to (1) with boundary condition  $g_o = 0$ , then  $g_i > g_{i^*}$  for each  $i \in V \setminus \{o\}$ .*

In Theorem 2.2 below, “inf sup” are used for the upper bounds of  $\lambda_p$ , e.g., each test function  $f \in \mathcal{F}_I$  produces a upper bound  $\sup_{i \in V \setminus \{o\}} I_i(f)^{-1}$ , so this part is called variational formula for upper estimates of  $\lambda_p$ . Dually, the “sup inf” are used for the lower estimates of  $\lambda_p$ . Among them, the ones expressed by operator  $R$  are easiest to compute in practice, and the ones expressed by  $II$  are hardest to compute but provide better estimates. Because of “inf sup”, a localizing procedure is used for the test function to avoid  $I(f) \equiv \infty$  for instance, which is removed out automatically for the “sup inf” part.

**Theorem 2.2** *The following variational formulas hold for  $\lambda_p$  defined by (3).*

(1) *Single summation forms*

$$\sup_{f \in \mathcal{F}_I} \inf_{i \in V \setminus \{o\}} I_i(f)^{-1} \leq \lambda_p \leq \inf_{f \in \mathcal{F}_I} \sup_{i \in V \setminus \{o\}} I_i(f)^{-1},$$

(2) *Double summation forms*

$$\sup_{f \in S(\mathcal{F})} \inf_{i \in V \setminus \{o\}} II_i(f)^{-1} \leq \lambda_p \leq \inf_{f \in S(\mathcal{F})} \sup_{i \in V \setminus \{o\}} II_i(f)^{-1}$$

with  $S(\mathcal{F}) = \mathcal{F}_\Pi$  or  $\mathcal{F}_I$  and  $S(\widetilde{\mathcal{F}}) = \widetilde{\mathcal{F}}_\Pi$ , or  $\widetilde{\mathcal{F}}_I$ , or  $\widetilde{\mathcal{F}}'_\Pi \cup \widetilde{\mathcal{F}}_\Pi$ .

(3) *Difference forms*

$$\sup_{w \in \mathcal{W}} \inf_{i \in V \setminus \{o\}} R_i(w) \leq \lambda_p \leq \inf_{w \in \mathcal{W}} \sup_{i \in V \setminus \{o\}} \widetilde{R}_i(w).$$

The six equalities in the three terms above hold once  $p \geq 2$ .

We write  $\tilde{\mu}(A) = \sum_{k \in A} \mu_k$  for some measure  $\mu$  and set  $A$ . Then

$$\mu(V_i) = \sum_{k \in V_i} \mu_k, \quad \hat{\nu}(\mathcal{P}(i)) = \sum_{k \in \mathcal{P}(i)} \hat{\nu}_k, \quad i \in V \setminus \{o\}.$$

Define

$$\sigma = \sup_{j \in V \setminus \{o\}} \mu(T_j) \hat{\nu}(\mathcal{P}(i)).$$

and

$$\#(A) = \text{number of elements in the set } A,$$

for some set  $A$ . As applications of Theorem 2.2, we have the following theorem.

**Theorem 2.3** *For  $p \in (1, \infty)$ , we have*

$$\sigma^{-1} \geq \lambda_p \geq \left[ \left( \hat{p}^{p-1} \sup_{i \in V \setminus \{o\}} (1 + (p-1)C_i) \right) \sigma \right]^{-1},$$

where

$$C_i = \#(J(i)) + \sum_{s \in J(i)} \sum_{k \in V_s} (\#(J(k)) - 1), \quad i \in V.$$

The theorem effectively presents us the quantitative estimates of the  $p$ -Laplacian Dirichlet eigenvalue on a tree with finite vertexes. For the degenerated case of the tree (only one branch), the results reduce to that on half line in [5].

**Example 2.4** *Let  $T$  be a  $r$  ( $r \geq 1$ ) order homogeneous tree (i.e.,  $\#(J(i)) = r$ ,  $\forall i \in V \setminus \{o\}$ ) with maximal level  $N$  ( $\leq \infty$ ) and root  $o$ , which has a child, i.e.,  $\#(J(o)) = 1$ . Assume that  $t \in (0, 1/r)$ ,  $\mu_k = t^{|k|}$  and  $\nu_k = at^{|k|}$  ( $a > 0$ ) for  $k \in V$ . For  $p \in (1, \infty)$ , denote*

$$B_p = \hat{p}^{p-1} \sup_{i \in V \setminus \{o\}} (1 + (p-1)C_i).$$

We have

$$\sigma^{-1} \geq \lambda_p \geq (B_p \sigma)^{-1},$$

where

$$B_p = \begin{cases} \hat{p}^{p-1} [1 + (p-1)(r^N + r - 2)], & r \geq 2, \\ p \hat{p}^{p-1}, & r = 1 \end{cases}$$

and

$$\sigma = \frac{1}{a(1-rt)(1-t^{\hat{p}-1})^{p-1}} \sup_{n \in [1, N+1)} \left\{ (1-(rt)^{N-n+1})(1-t^{n(\hat{p}-1)})^{p-1} \right\}.$$

If  $N = \infty$ , then

$$\sigma = \frac{1}{a(1-rt)(1-t^{\hat{p}-1})^{p-1}}.$$

### 3 Proofs of the main results

Without loss of generality, we assume that the root  $o$  has only one child (i.e.,  $\#(J(o)) = 1$ ), the level counting begins from the child of the root  $o$  (i.e.,  $|o| = 0$  and  $|J(o)| = 1$ ). For convenience, we write 1 as the unique child of root  $o$  in the proofs of Lemma 2.1, i.e.,  $J(o) = \{1\}$  and  $\mathcal{P}(1) = \{1\}$ .

**Proof of Lemma 2.1** We prove the theorem by dividing it into two steps as follows.

(1) we prove that  $g_o = 0 \neq g_1$ .

If  $g_1 = 0$ , then  $\Omega_p g(1) = -\lambda_p \mu_1 |g_1|^{p-2} g_1 = 0$ , and

$$\Omega_p g(1) = \sum_{j \in J(1)} \nu_j |g_j|^{p-2} g_j.$$

Therefore,

$$\sum_{j \in J(1)} \nu_j |g_j|^{p-2} g_j = 0. \quad (4)$$

Moreover,  $g_j = 0$  for  $j \in J(1)$ , which will be proved as follows.

Let  $A = \{j \in J(1) : g_j < 0\}$ ,  $B = \{j \in J(1) : g_j \geq 0\}$  and  $C_0 = \{j \in J(1) : g_j = 0\}$ . Then we prove that  $A = B \setminus C_0 = \emptyset$ , which is sufficient to show that  $A = \emptyset$  by (4). We prove that  $A = \emptyset$  by making a contradiction. If  $A \neq \emptyset$ , then define function  $\tilde{g}$  on  $T$  satisfying  $\tilde{g}_o = 0$ ,  $\tilde{g}_1 = x > 0$ , and

$$\tilde{g}_i = \begin{cases} -g_i, & i \in V_A, \\ g_i, & i \in V_B. \end{cases}$$

where  $V_C := \cup_{i \in C} V_i$  for some set  $C$ . Then

$$\begin{aligned} D_p(\tilde{g}, \tilde{g}) &= \sum_{j \in V \setminus \{o\}} \nu_j |\tilde{g}_j - \tilde{g}_{j^*}|^p \\ &= \nu_1 |\tilde{g}_1 - \tilde{g}_o|^p + \sum_{j \in A} \nu_j |\tilde{g}_j - \tilde{g}_{j^*}|^p + \sum_{j \in B} \nu_j |\tilde{g}_j - \tilde{g}_{j^*}|^p + \\ &\quad + \sum_{j \in V_A \setminus A} \nu_j |g_j - g_{j^*}|^p + \sum_{j \in V_B \setminus B} \nu_j |g_j - g_{j^*}|^p \\ &= \nu_1 |x|^p + \sum_{j \in A} \nu_j |g_j - x|^p + \sum_{j \in B} \nu_j |g_j - x|^p + \\ &\quad + \sum_{j \in V_A \setminus A} \nu_j |g_j - g_{j^*}|^p + \sum_{j \in V_B \setminus B} \nu_j |g_j - g_{j^*}|^p \\ D_p(g, g) &= \sum_{j \in V \setminus \{o\}} \nu_j |g_j - g_{j^*}|^p \\ &= \sum_{j \in A} \nu_j |g_j|^p + \sum_{j \in B} \nu_j |g_j|^p + \sum_{j \in V_{A \cup B} \setminus (A \cup B)} \nu_j |g_j - g_{j^*}|^p \\ &\quad (\text{by } g_1 = g_o = 0). \end{aligned}$$

Therefore,

$$\begin{aligned} D_p(\tilde{g}, \tilde{g}) &= D_p(g, g) + \nu_1 |x|^p + \sum_{j \in A} \nu_j (|g_j + x|^p - |g_j|^p) + \sum_{j \in B} \nu_j (|g_j - x|^p - |g_j|^p) \\ &= D_p(g, g) + \left\{ (\nu_1 + \sum_{j \in C_0} \nu_j) |x|^p + \sum_{j \in A} \nu_j (|g_j + x|^p - |g_j|^p) + \sum_{j \in B \setminus C_0} \nu_j (|g_j - x|^p - |g_j|^p) \right\} \\ &=: D_p(g, g) + \mu_1 G(x). \end{aligned}$$

We will see that there exists  $x > 0$  such that  $G(x) < 0$ . Indeed, let  $\ell = \min\{|g_j|/2 : j \in A \cup (B \setminus C_0)\}$ . Then  $\ell > 0$  by  $d_1 < \infty$ . For  $p > 0$  and  $x \in (0, \ell)$ , we have  $|g_j + x|^p < |g_j|^p$  if  $j \in A$  and  $|g_j - x|^p < |g_j|^p$  if  $j \in B \setminus C_0$ . Since  $pa^{p-1}(b-a) < b^p - a^p < pb^{p-1}(b-a)$  provided  $0 < a < b$  and  $p \geq 1$ , for  $j \in A$  and

$x \in (0, \ell)$ , we have

$$\begin{aligned}
 |g_j|^p - |g_j + x|^p &\geq p|g_j + x|^{p-1}(|g_j| - |g_j + x|) \\
 &= p|g_j + x|^{p-1}(-g_j - (-g_j - x)) \\
 &= p|g_j + x|^{p-1}x \\
 &\geq \inf_{x \in (0, \ell)} \left\{ \frac{|g_j + x|^{p-1}}{x^{p-2}} \right\} p x^{p-1}.
 \end{aligned}$$

For  $j \in B_0 \setminus C$  and  $x \in (0, \ell)$ , we have

$$\begin{aligned}
 |g_j|^p - |g_j - x|^p &\geq p|g_j - x|^{p-1}(|g_j| - |g_j - x|) \\
 &= p|g_j - x|^{p-1}(g_j - (g_j - x)) \\
 &= p|g_j - x|^{p-1}x \\
 &\geq \inf_{x \in (0, \ell)} \left\{ \frac{|g_j - x|^{p-1}}{x^{p-2}} \right\} p x^{p-1}.
 \end{aligned}$$

When  $p \geq 2$ , we have

$$\inf_{x \in (0, \ell)} \left\{ \frac{|g_j + x|^{p-1}}{x^{p-2}} \right\} \geq \inf_{x \in (0, |g_j|/2)} \left\{ \frac{|g_j + x|^{p-1}}{x^{p-2}} \right\} \geq \frac{|g_j|}{2} \quad \text{for } j \in A,$$

and

$$\inf_{x \in (0, \ell)} \left\{ \frac{|g_j - x|^{p-1}}{x^{p-2}} \right\} \geq \inf_{x \in (0, |g_j|/2)} \left\{ \frac{|g_j - x|^{p-1}}{x^{p-2}} \right\} \geq \frac{|g_j|}{2} \quad \text{for } j \in B \setminus C_0.$$

So

$$\begin{aligned}
 &\sum_{j \in A} v_j(|g_j + x|^p - |g_j|^p) + \sum_{j \in B \setminus C_0} v_j(|g_j - x|^p - |g_j|^p) \\
 &\leq -\frac{p}{2} x^{p-1} \left\{ \sum_{j \in A} v_j |g_j| + \sum_{j \in B \setminus C_0} v_j |g_j| \right\} \\
 &=: -\frac{p}{2} G_0 x^{p-1} < 0 \quad (p \geq 2).
 \end{aligned}$$

Hence,

$$G(x) < (v_1 + \sum_{j \in C_0} v_j) x^p - \frac{p}{2} G_0 x^{p-1}.$$

Let  $0 < x < \min \{pG_0[2(v_1 + \sum_{j \in C_0} v_j)]^{-1}, |g_j|/2, j \in A \cup (B \setminus C_0)\}$ . Then  $G(x) < 0$ . Moreover,

$$D_p(\tilde{g}, \tilde{g}) \leq D_p(g, g), \quad p \geq 2.$$

Since

$$\mu(|\tilde{g}|^p) = \sum_{j \in V} \mu_j \tilde{g}_j = \mu_1 x^p + \mu(|g|^p) > \mu(|g|^p),$$

and  $g \in L^p(\mu)$ , we have

$$\frac{D_p(\tilde{g}, \tilde{g})}{\mu(|\tilde{g}|^p)} < \frac{D_p(g, g)}{\mu(|g|^p)} \leq \lambda_p,$$

which is a contradiction with (3). Therefore,  $A = \emptyset$ , and  $g_j = 0$  for  $j \in J(1)$ . By the induction, we have  $g_i = 0$  for  $i \in V \setminus 0$ . Hence, we must have  $g_1 \neq 0$ .

(2) We prove that the eigenfunction satisfies  $g_{i^*} < g_i$  for  $i \in V \setminus \{o\}$ .

We prove the result by making a contradiction. Since  $g_1 \neq g_o = 0$ , without loss of generality, assume that  $g_1 > 0 = g_o$  (otherwise, replace  $g$  by  $-g$ , which is also an eigenfunction of  $\lambda_p$ ). If there exists  $a \in V \setminus \{o\}$  satisfying  $0 = g_o < g_1 < \dots < g_a \geq g_b$  for some  $b \in J(a)$  ( $\mathcal{P}(b) = \{1, \dots, a, b\}$  and their levels satisfy  $|o| \leq |1| \leq \dots \leq |a| \leq |b|$ ), then set

$$\bar{g}_i = \begin{cases} g_i, & i \notin V_b, \\ g_a, & i \in V_b. \end{cases}$$

We have

$$\begin{aligned}\Omega_p \bar{g}(k) &= \sum_{j \in J(k)} \nu_j |\bar{g}_j - \bar{g}_k|^{p-2} (\bar{g}_j - \bar{g}_k) + \nu_k |\bar{g}_{k^*} - \bar{g}_k|^{p-2} (\bar{g}_{k^*} - \bar{g}_k) \\ &= \begin{cases} 0, & k \in V_b; \\ \Omega_p g(k), & k \notin V_b, k \neq a; \\ \sum_{j \in J(a), j \neq b} \nu_j |g_j - g_a|^{p-2} (g_j - g_a) + \nu_a |g_{a^*} - g_a|^{p-2} (g_{a^*} - g_a), & k = a, \end{cases} \\ &= \begin{cases} 0, & k \in V_b; \\ \Omega_p g(k), & k \notin V_b, k \neq a; \\ \Omega_p g(a) - \nu_b |g_b - g_a|^{p-2} (g_b - g_a), & k = a, \end{cases}\end{aligned}$$

and

$$\begin{aligned}D_p(\bar{g}, \bar{g}) &= (-\Omega_p \bar{g}, \bar{g})_\mu = - \sum_{k \in V \setminus \{0\}} \mu_k \bar{g}_k \Omega_p \bar{g}(k) \\ &= - \sum_{k \in V_b} \mu_k \bar{g}_k \Omega_p \bar{g}(k) - \sum_{k \notin V_b, k \neq a} \mu_k \bar{g}_k \Omega_p \bar{g}(k) - \mu_a \bar{g}_a \Omega_p \bar{g}(a) \\ &= - \sum_{k \notin V_b, k \neq a} \mu_k g_k \Omega_p g(k) - \mu_a g_a \Omega_p g(a).\end{aligned}$$

By assumption  $g_a \geq g_b$ , we have

$$\Omega_p \bar{g}(a) = \Omega_p g(a) - \nu_b |g_b - g_a|^{p-2} (g_b - g_a) \geq \Omega_p g(a).$$

Moreover,

$$\begin{aligned}D_p(\bar{g}, \bar{g}) &\leq - \sum_{k \notin V_b, k \neq a} \mu_k g_k \Omega_p g(k) - \mu_a g_a \Omega_p g(a) \\ &= - \sum_{k \notin V_b} \mu_k g_k \Omega_p g(k) \\ &= \lambda_p \sum_{k \notin V_b} \mu_k |g_k|^p.\end{aligned}$$

Since  $b \notin \mathcal{P}(1)$ , by definition of  $\lambda_p$ , we have

$$\lambda_p \leq \frac{D_p(\bar{g}, \bar{g})}{\mu(|\bar{g}|^p)} \leq \frac{\lambda_p \sum_{k \notin V_b} \mu_k |g_k|^p}{\sum_{k \notin V_b} \mu_k |g_k|^p + \sum_{k \in V_b} \mu_k |g_a|^p} < \lambda_p$$

once  $\lambda_p > 0$ , which is a contradiction. Therefore,  $g_b > g_a$  for each  $b \in J(a)$ .  $\square$

Obviously, on the setting of a finite tree  $T$ , the eigenfunction  $g$  of the  $p$ -Laplacian Dirichlet eigenvalue satisfies  $g_i > g_{i^*}$  for every  $i \in V$ . Before moving further, we introduce a general equation and discuss the origin of operators used in Theorem 2.2. Recall that  $\Lambda_m = \{i : |i| = m\}$  and  $N$  is the maximal level of tree  $T$ . Define

$$V(n) = \cup_{m=0}^n \Lambda_m.$$

Consider

$$\text{Poisson equation : } \quad \Omega_p g(i) = -\mu_i |f_i|^{p-2} f_i, \quad i \in V \setminus \{o\}.$$

By multiplying  $\mu_i$  on both sides of the equation and making summation with respect to  $i \in V_k \cap V(n)$  for some  $k \in V \setminus \{o\}$  with  $|k| \leq n$ , it is easy to check that

$$\sum_{j \in \Lambda_{n+1} \cap V_k} \nu_j |g_{j^*} - g_j|^{p-2} (g_{j^*} - g_j) - \nu_k |g_k - g_{k^*}|^{p-2} (g_{k^*} - g_k) = \sum_{j \in V_k \cap V(n)} \mu_j |f_j|^{p-2} f_j, \quad |k| \leq n. \quad (5)$$

If  $\lim_{n \rightarrow N} \sum_{j \in \Lambda_{n+1} \cap V_k} \nu_j |g_{j^*} - g_j|^{p-2} (g_{j^*} - g_j) = 0$  (which is obvious for  $N < \infty$ ), then we obtain the form of the operator  $I$  by letting  $n \rightarrow N$  and  $f = \lambda_p^{\frac{p-1}{p}} g$  in (5). Moreover, if  $g_o = 0$  and  $g_i > g_{i^*}$  for  $i \in V \setminus \{o\}$ , then

$$g_i = \sum_{k \in \mathcal{P}(i)} \hat{\nu}_k \left( \sum_{j \in V_k} \mu_j |f_j|^{p-2} f_j \right)^{\frac{p-1}{p}}.$$

This explains where the operator  $\Pi$  comes from. Similarly, from the eigenequation (1), we obtain the operator  $R$  by letting  $w_i = g_i/g_{i^*}$ . The eigenequation is a “bridge” among these operators. Let

$$\tilde{\lambda}_p = \inf\{D_p(f) : \mu(|f|^p) = 1, \exists 1 \leq n < N+1 \text{ such that } f_i = f_{i^*} \text{ for } |i| \geq n+1\}.$$

If  $\sum_{k \in V} \mu_k < \infty$ , then  $\lambda_p = \tilde{\lambda}_p$  as will be seen in Lemma 3.1 below. To this end, define

$$\lambda_p^{(m)} = \inf\{D_p(f) : \mu(|f|^p) = 1, f_i = f_{i^*} \text{ for } |i| \geq m+1\}, \quad 1 \leq m < N+1.$$

Let

$$\begin{aligned} \tilde{\mu}_i &= \mu_i, \quad \tilde{\nu}_i = \nu_i & \text{for } |i| \leq m-1 \text{ and } |j| \leq m-1; \\ \tilde{\mu}_i &= \sum_{j \in V_i} \mu_j, \quad \tilde{\nu}_i = \nu_i & \text{for } |i| = m. \end{aligned}$$

For  $f$  with  $f_i = f_{i^*}$  for  $|i| \geq m+1$ , we have

$$D_p(f) = \sum_{i \in V(m) \setminus \{o\}} \tilde{\nu}_i |f_i - f_{i^*}|^p =: \tilde{D}_p(f), \quad \mu(|f|^p) = \sum_{i \in V(m)} \tilde{\mu}_i |f_i|^p =: \tilde{\mu}(|f|^p).$$

So  $\lambda_p^{(m)}$  is  $p$ -Laplacian eigenvalue of the local Dirichlet form  $(\tilde{D}, \mathcal{D}(\tilde{D}))$  with state space  $T(m)$ , which is a finite tree with maximal level  $m$  and coincides with tree  $T$  restricted to the first  $m-1$  levels.

This following lemma presents us an approximating procedure, which guarantees that some properties hold obviously once that hold for finite cases (see Step 4 in proof of Theorem 2.2 below). For simplicity, we use “iff” to denote “if and only if” and  $\uparrow$  (resp.  $\downarrow$ ) to denote increasing and decreasing throughout the paper.

**Lemma 3.1** Assume that  $\sum_{k \in V} \mu_k < \infty$  (i.e.,  $\mu(T) < \infty$ ). We have  $\lambda_p = \tilde{\lambda}_p$  and  $\lambda_p^{(n)} \downarrow \lambda_p$  as  $n \rightarrow N$ .

*Proof* By definition of  $\lambda_p$ , for any  $\varepsilon > 0$ , there exists  $f$  such that  $D_p(f)/\mu(|f|^p) \leq \lambda_p + \varepsilon$ . Construct  $f^{(n)}$  such that  $f_i^{(n)} = f_i$  for  $|i| \leq n$  and  $f_i^{(n)} = f_{i^*}$  for  $|i| \geq n+1$ . Since  $\sum_{k \in V} \mu_k < \infty$ , we have

$$\begin{aligned} D_p(f^{(n)}) &= \sum_{i \in V \setminus \{o\}} \nu_i |f_i^{(n)} - f_{i^*}^{(n)}|^p = \sum_{i \in V(n) \setminus \{o\}} \nu_i |f_i - f_{i^*}|^p \uparrow D_p(f), \quad n \rightarrow N, \\ \mu(|f^{(n)}|^p) &= \sum_{i \in V(n) \setminus \{o\}} \mu_i |f_i|^p + \sum_{i \in \Lambda_{n+1}} \mu(V_i) |f_{i^*}|^p \uparrow \mu(|f|^p) \quad n \rightarrow N. \end{aligned}$$

By definitions of  $\lambda_p$ ,  $\tilde{\lambda}_p$  and  $\lambda_p^{(n)}$ , the required assertion holds.  $\square$

Using the similar methods introduced in [3], there are not much difficulties to complete the proof of Theorem 2.2. Therefore, we will present more details of the proofs of Theorem 2.3 but only some keys for that of Theorem 2.2 in the following.

**Proof of Theorem 2.2** We adopt the following circle to prove the upper bounds of  $\lambda_p$ .

$$\begin{aligned} \lambda_p &\leq \inf_{f \in \mathcal{F}_H' \cup \mathcal{F}_H} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1} \\ &\leq \inf_{f \in \mathcal{F}_H} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1} = \inf_{f \in \mathcal{F}_I} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1} = \inf_{f \in \mathcal{F}_I} \sup_{i \in V \setminus \{o\}} I_i(f)^{-1} \\ &\leq \inf_{w \in \mathcal{W}} \sup_{i \in V \setminus \{o\}} \tilde{R}_i(w) \leq \lambda_p. \end{aligned}$$

The second inequality above is clear and the remainders are proved by several steps as follows.

**Step 1** Prove that  $\lambda_p \leq \inf_{f \in \mathcal{F}_H' \cup \mathcal{F}_H} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1}$ .

For  $f \in \mathcal{F}_H'$ , there exists  $n \in E$  such that  $f_i = f_{i^*}$  for  $|i| \geq n+1$ . Let

$$g_i = \sum_{k \in \mathcal{D}(i)} \left( \frac{1}{\nu_k} \sum_{j \in V_k} \mu_j f_j^{p-1} \right)^{\hat{p}-1}, \quad |i| \leq n$$

and  $g_i = g_{i^*}$  for  $|i| \geq n+1$ . Then  $g \in L^p(\mu)$ ,  $g_i = f_i \Pi_i(f)^{\hat{p}-1}$  for  $|i| \leq n$  and  $f_i \neq 0$ . Moreover,

$$g_i - g_{i^*} = \left( \frac{1}{\nu_i} \sum_{j \in V_i} \mu_j f_j^{p-1} \mathbf{1}_{\{|i| \leq n\}} \right)^{\hat{p}-1}.$$



Inserting this term into  $D_p(g)$ , we have

$$\begin{aligned}
D_p(g) &= \sum_{j \in V \setminus \{o\}} (g_j - g_{j^*}) \sum_{k \in V_j} \mu_k f_k^{p-1} \mathbf{1}_{\{|j| \leq n\}} \\
&= \sum_{k \in V \setminus \{o\}} \mu_k f_k^{p-1} \sum_{j \in \mathcal{P}(k)} \mathbf{1}_{\{|j| \leq n\}} (g_j - g_{j^*}) \quad (\text{since } k \in V_j \text{ iff } j \in \mathcal{P}(k)) \\
&= \sum_{k \in V \setminus \{o\}} \mu_k f_k^{p-1} g_k \quad (\text{since } g_i = g_{i^*} \text{ for } |i| \geq n+1).
\end{aligned}$$

Since  $g \in L^p(\mu)$ , we further obtain

$$D_p(g) \leq \sum_{k \in V \setminus \{o\}} \mu_k |g_k|^p \sup_{k \in V \setminus \{o\}} \left( \frac{f_k}{g_k} \right)^{p-1} \leq \mu(|g|^p) \sup_{k \in V \setminus \{o\}} \Pi_k(f)^{-1}.$$

Hence,

$$\lambda_p \leq \frac{D_p(g)}{\mu(|g|^p)} \leq \sup_{k \in V \setminus \{o\}} \Pi_k(f)^{-1}.$$

The inequality also holds for  $f \in \widetilde{\mathcal{F}}'_\Pi$  since the key point in its proof is  $g = f\Pi(f) \in L^p(\mu)$ , which also holds for  $f \in \widetilde{\mathcal{F}}'_\Pi$ . So the required assertion holds.

Step 2 Prove that

$$\inf_{f \in \widetilde{\mathcal{F}}_\Pi} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1} = \inf_{f \in \widetilde{\mathcal{F}}_I} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1} = \inf_{f \in \widetilde{\mathcal{F}}_I} \sup_{i \in V \setminus \{o\}} I_i(f)^{-1}.$$

(a) We first prove that

$$\inf_{f \in \widetilde{\mathcal{F}}_\Pi} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1} \leq \inf_{f \in \widetilde{\mathcal{F}}_I} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1} \leq \inf_{f \in \widetilde{\mathcal{F}}_I} \sup_{i \in V \setminus \{o\}} I_i(f)^{-1}.$$

Since  $\widetilde{\mathcal{F}}_I \subset \widetilde{\mathcal{F}}_\Pi$ , the first inequality is clear. For  $f \in \widetilde{\mathcal{F}}_I$ , there exists  $1 \leq n < N+1$  such that  $f_i = f_{i^*}$  for  $|i| \geq n+1$  and  $f_i > f_{i^*}$  for  $|i| \leq n$ . Since  $f_i = \sum_{k \in \mathcal{P}(i)} (f_k - f_{k^*})$  for  $|i| \leq n$ , inserting this term into the denominator of  $\Pi_i(f)$  and using the proportional property, we have

$$\inf_{i \in V \setminus \{o\}} \Pi_i(f) = \inf_{i \in V(n) \setminus \{o\}} \Pi_i(f) \geq \inf_{i \in V \setminus \{o\}} I_i(f).$$

and the required assertion holds since  $f \in \widetilde{\mathcal{F}}_I$  is arbitrary.

(b) Prove the equality.

For  $f \in \widetilde{\mathcal{F}}_\Pi$ ,  $\exists n \in [1, N+1)$  such that  $f_i = f_{i^*}$  for  $|i| \geq n+1$  and  $f \neq 0$ . Let

$$g_i = \sum_{k \in \mathcal{P}(i)} \left( \frac{1}{v_k} \sum_{j \in V_k} \mu_j f_j^{p-1} \right)^{\hat{p}-1}, \quad 0 < |i| \leq n,$$

$g_o = 0$  and  $g_i = g_{i^*}$  for  $|i| \geq n+1$ . Then  $g \in \widetilde{\mathcal{F}}_I$  and

$$g_i - g_{i^*} = \left( \frac{1}{v_i} \sum_{j \in V_i} \mu_j f_j^{p-1} \right)^{\hat{p}-1}, \quad 0 < |i| \leq n.$$

Moreover,

$$v_i (g_i - g_{i^*})^{p-1} \leq \sum_{j \in V_i} \mu_j g_j^{p-1} \sup_{j \in V_i} \left( \frac{f_j}{g_j} \right)^{p-1} = \sum_{j \in V_i} \mu_j g_j^{p-1} \sup_{j \in V_i} \Pi_i(f)^{-1}, \quad i \in V \setminus \{o\}.$$

Hence,

$$\sup_{k \in V \setminus \{o\}} I_k(g)^{-1} \leq \sup_{k \in V \setminus \{o\}} \Pi_k(f)^{-1}.$$

Then the assertion follows by making the infimum over  $\widetilde{\mathcal{F}}_I$  first and then the infimum over  $\widetilde{\mathcal{F}}_\Pi$ .

Step 3 Prove that  $\inf_{f \in \widetilde{\mathcal{F}}_\Pi} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1} \leq \inf_{w \in \widetilde{\mathcal{W}}} \sup_{i \in V \setminus \{o\}} \widetilde{R}_i(w)$ .

First, we change the form of  $\widetilde{R}$ . For  $w \in \widetilde{\mathcal{W}}$  with  $w_i = 1$  for  $|i| \geq m+1$ , let  $g$  be a positive function on  $V \setminus \{o\}$  with  $g_o = 0$  such that  $w_i = g_i/g_{i^*}$ . Applying Lemma 2.1 to the finite tree  $T(m)$ , we have  $g_i > g_{i^*}$  for  $|i| \leq m$  and  $g_i = g_{i^*}$  for  $|i| \geq m+1$ . Since

$$\sum_{j \in J(i)} v_j(w_j - 1)^{p-1} < v_i(1 - w_i^{-1})^{p-1} \quad \text{for } |i| \leq m,$$

we have  $\widetilde{R}_i(w) = -\widetilde{\Omega}_p g(i)/\mu_i g_i^{p-1} > 0$  for  $|i| \leq m$  and  $\widetilde{R}_i(w) = 0$  for  $|i| \geq m+1$ , where  $\widetilde{\Omega}$  is a change form of  $\Omega$  with  $\mu_i$  replaced by  $\tilde{\mu}_i$  for  $|i| \leq m$ .

Now, we come back to the main assertion. For  $w \in \widetilde{\mathcal{W}}$  with  $w_i = 1$  for  $|i| \geq m+1$ , let  $g$  be the function mentioned above and

$$f_i^{p-1} = \begin{cases} -\mu_i^{-1} \left[ \sum_{j \in J(i)} v_j(g_j - g_i)^{p-1} + v_i(g_i - g_{i^*})^{p-1} \right], & |i| \leq m-1, \\ \tilde{\mu}_i^{-1} v_i(g_i - g_{i^*})^{p-1}, & |i| = m, \\ f_{i^*}^{p-1}, & |i| \geq m+1. \end{cases}$$

Then  $\mu_i f_i^{p-1} = -\widetilde{\Omega}_p g(i) > 0$  for  $|i| \leq m$ . By (5), we have

$$v_k(g_k - g_{k^*})^{p-1} = \sum_{j \in V_k \cap V(m-1)} \mu_j f_j^{p-1} + \sum_{j \in \Lambda_m \cap V_k} v_j(g_j - g_{j^*})^{p-1}, \quad |k| \leq m-1. \quad (6)$$

Since

$$v_j(g_j - g_{j^*})^{p-1} = \sum_{i \in V_j} \mu_i f_i^{p-1} = \sum_{i \in V_j} \mu_i f_i^{p-1}, \quad |j| = m,$$

we have

$$\begin{aligned} \sum_{j \in \Lambda_m \cap V_k} v_j(g_j - g_{j^*})^{p-1} &= \sum_{j \in \Lambda_m \cap V_k} \sum_{i \in V_j} \mu_i f_i^{p-1} = \sum_{j \in (T \setminus V(m-1)) \cap V_k} \mu_j f_j^{p-1} \\ &= \sum_{j \in V_k} \mu_j f_j^{p-1} - \sum_{j \in V(m-1) \cap V_k} \mu_j f_j^{p-1}, \quad |k| \leq m. \end{aligned}$$

Inserting this term into (6), we arrived at

$$\begin{aligned} v_k(g_k - g_{k^*})^{p-1} &= \sum_{j \in V_k \cap V(m-1)} \mu_j f_j^{p-1} + \sum_{j \in \Lambda_m \cap V_k} v_j(g_j - g_{j^*})^{p-1} \\ &= \sum_{j \in V_k} \mu_j f_j^{p-1}, \quad 0 < |k| \leq m-1. \end{aligned}$$

Hence,

$$v_k(g_k - g_{k^*})^{p-1} = \sum_{j \in V_k} \mu_j f_j^{p-1} \quad 0 < |k| \leq m.$$

Moreover,

$$g_i = \sum_{k \in \mathcal{P}(i)} \left( \frac{1}{v_k} \sum_{j \in V_k} \mu_j f_j^{p-1} \right)^{\frac{1}{p-1}} \quad 0 < |i| \leq m,$$

and  $\widetilde{R}_i(w) = (f_i/g_i)^{p-1} = \Pi_i(f)^{-1}$  for  $0 < |i| \leq m$ . Since  $\widetilde{R}_i(w) = 0$  and  $f_i = f_{i^*}$  for  $|i| \geq m+1$ , we obtain

$$\sup_{i \in V \setminus \{o\}} \widetilde{R}_i(w) = \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1} \geq \inf_{f \in \widetilde{\mathcal{F}}_{\Pi}} \sup_{i \in V \setminus \{o\}} \Pi_i(f)^{-1}, \quad w \in \widetilde{\mathcal{W}},$$

and the required assertion holds.

Step 4 Prove that  $\inf_{w \in \widetilde{\mathcal{W}}} \sup_{i \in V \setminus \{o\}} \widetilde{R}_i(w) \leq \lambda_p$  once  $p \geq 2$ .

Let  $g$  with  $g_o = 0$  be an eigenfunction of local  $p$ -Laplacian eigenvalue  $\lambda_p^{(m)}$  and extend  $g$  to  $V \setminus \{o\}$  by setting  $g_i = g_{i^*}$  for  $|i| \geq m+1$ . Put  $w_i = g_i/g_{i^*}$  for  $i \in V \setminus \{o\}$ . Then  $w \in \widetilde{\mathcal{W}}$  provided  $p \geq 2$ .

Since  $m < \infty$ , we have  $\widetilde{R}_i(w) = \lambda_p^{(m)} > 0$  for  $i \in V(m) \setminus \{o\}$  and  $\widetilde{R}_i(w) = 0$  for  $T \setminus V(m)$ . Therefore,

$$\begin{aligned} \lambda_p^{(m)} &= \sup_{i \in V \setminus \{o\}} \widetilde{R}_i(w) \\ &\geq \inf_{w \in \widetilde{\mathcal{W}}: w_i=1 \text{ for } |i| \geq m+1} \sup_{i \in V(m) \setminus \{o\}} \widetilde{R}_i(w) \\ &\geq \inf_{w \in \widetilde{\mathcal{W}}: \exists n \geq 1 \text{ such that } w_i=1 \text{ for } |i| \geq n+1} \sup_{i \in V \setminus \{o\}} \widetilde{R}_i(w) \\ &\geq \inf_{w \in \widetilde{\mathcal{W}}} \sup_{i \in V \setminus \{o\}} \widetilde{R}_i(w). \end{aligned}$$

Letting  $m \rightarrow N$ , the assertion then follows by Lemma 3.1.

By now, we have finished the proof of the estimates of upper  $\lambda_p$ . Dually, one can prove the lower estimates without too much difficulty. We ignore the details here.  $\square$

Define  $T_{i,j} = T_i \cup T_j$ , correspondingly,  $V_{i,j} = V_i \cup V_j$ . Then

$$V_{J(i)} = \{k : s \in J(i) \text{ and } k \in V_s\}.$$

Similarly,

$$J(V_i) = \{k : s \in V_i \text{ and } k \in J(s)\}.$$

It is obvious that  $J(V_i) = V_{J(i)}$ . Without loss of generality, we adopt convention that  $\mu(V_k) = 0$  if  $V_k = \phi$ . We use notation  $|J(i)|$  indicating the level of  $J(i)$ , i.e.,  $|J(i)| = |i| + 1$ . For simplicity, we also write  $\varphi_i = \hat{\nu}(\mathcal{P}(i))^{p-1}$  in the proof of Theorem 2.3.

**Proof of Theorem 2.3** First, we prove that  $\lambda_p^{-1} \leq (\hat{p}^{p-1} \sup_{i \in V \setminus \{o\}} (1 + (p-1)C_i)) \sigma$ . By calculation, we have

$$\begin{aligned} \sum_{j \in V_i} \mu_j f_j^{p-1} &= \sum_{j \in V_i} f_j^{p-1} \left[ \mu(V_j) - \sum_{k \in J(j)} \mu(V_k) \right] \\ &= \sum_{j \in V_i} \mu(V_j) f_j^{p-1} - \sum_{j \in V_i} \sum_{k \in J(j)} \mu(V_k) f_j^{p-1} \\ &= \sum_{j \in V_i} \mu(V_j) f_j^{p-1} - \sum_{k \in V_{J(i)}} \mu(V_k) f_{k^*}^{p-1} \quad (\text{since } J(V_i) = V_{J(i)}) \\ &= \mu(V_i) f_i^{p-1} + \sum_{k \in V_{J(i)}} \mu(V_k) (f_k^{p-1} - f_{k^*}^{p-1}) \quad (\text{since } V_i = \{i\} \cup V_{J(i)}). \end{aligned}$$

Put  $f_j = \varphi_j^{1/p}$  for  $j \in V \setminus \{o\}$ . Then

$$\begin{aligned} \sum_{j \in V_i} \mu_j f_j^{p-1} &= \mu(V_i) \varphi_i^{1/\hat{p}} + \sum_{k \in V_{J(i)}} \mu(V_k) (\varphi_k^{1/\hat{p}} - \varphi_{k^*}^{1/\hat{p}}) \\ &\leq \sigma \left[ \varphi_i^{-1/p} + \sum_{k \in V_{J(i)}} \frac{1}{\varphi_k} (\varphi_k^{1/\hat{p}} - \varphi_{k^*}^{1/\hat{p}}) \right]. \end{aligned}$$

Since  $\varphi_k \geq \varphi_{k^*}$ , we obtain

$$\sum_{k \in V_{J(i)}} \frac{1}{\varphi_k} (\varphi_k^{1/\hat{p}} - \varphi_{k^*}^{1/\hat{p}}) \leq (p-1) \sum_{k \in V_{J(i)}} (\varphi_{k^*}^{-1/p} - \varphi_k^{-1/p}). \quad (7)$$

Indeed, it suffices to show that

$$\varphi_k^{1/\hat{p}} - \varphi_{k^*}^{1/\hat{p}} \leq (p-1) \varphi_k (\varphi_{k^*}^{-1/p} - \varphi_k^{-1/p}),$$

or equivalently,

$$p \varphi_k^{1/\hat{p}} - \varphi_{k^*}^{1/\hat{p}} \leq (p-1) \varphi_k \varphi_{k^*}^{-1/p},$$

i.e.

$$p \varphi_k^{1/\hat{p}} \varphi_{k^*}^{1/p} \leq \varphi_{k^*} + (p-1) \varphi_k = p \left( \frac{1}{p} (\varphi_{k^*}^{1/p})^p + \frac{1}{\hat{p}} (\varphi_k^{1/\hat{p}})^{\hat{p}} \right),$$

which is obvious by Young's inequality.

Noticing that  $V_{J(i)} = J(V_i)$  and  $k \in J(j)$  if and only if  $k^* = j$ , we have

$$\sum_{k \in V_{J(i)}} \varphi_{k^*}^{-1/p} = \sum_{k \in J(V_i)} \varphi_{k^*}^{-1/p} = \sum_{j \in V_i} \sum_{k \in J(j)} \varphi_j^{-1/p} = \sum_{j \in V_i} \#(J(j)) \varphi_j^{-1/p},$$

Inserting the term to the inequality (7), we get

$$\begin{aligned} \sum_{k \in V_{J(i)}} \frac{1}{\varphi_k^{1/p}} (\varphi_k^{1/\hat{p}} - \varphi_{k^*}^{1/\hat{p}}) &\leq (p-1) \left\{ \sum_{j \in V_i} \#(J(j)) \varphi_j^{-1/p} - \sum_{k \in V_{J(i)}} \varphi_k^{-1/p} \right\} \\ &= (p-1) \left\{ \#(J(i)) \varphi_i^{-1/p} + \sum_{k \in V_{J(i)}} (|J(k)| - 1) \varphi_k^{-1/p} \right\} \\ &\leq (p-1) \left[ \#(J(i)) + \sum_{k \in V_{J(i)}} (|J(k)| - 1) \right] \varphi_i^{-1/p} \quad (\text{since } \varphi_k \geq \varphi_{k^*}). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j \in V_i} \mu_j \varphi_j^{1/p} &\leq \left[ 1 + (p-1) \left( \#(J(i)) + \sum_{s \in J(i)} \sum_{k \in V_s} (|J(k)| - 1) \right) \right] \sigma \varphi_i^{-1/p} \\ &= (1 + (p-1)C_i) \sigma \varphi_i^{-1/p}. \end{aligned}$$

Since

$$(\varphi_i^{\hat{p}-1} - \varphi_{i^*}^{\hat{p}-1})^{p-1} = \frac{1}{v_i} \quad \text{and} \quad \varphi_i^{1/[p(p-1)]} \varphi_{i^*}^{1/p} \leq \frac{1}{p} \varphi_i^{\hat{p}-1} + \frac{1}{\hat{p}} \varphi_{i^*}^{\hat{p}-1},$$

we obtain

$$\begin{aligned} I_i(\varphi^{1/p}) &= \frac{1}{v_i(\varphi_i^{1/p} - \varphi_{i^*}^{1/p})^{p-1}} \sum_{j \in V_i} \mu_j \varphi_j^{1/\hat{p}} \\ &\leq (1 + (p-1)C_i) \sigma \varphi_i^{-1/p} \left( \frac{\varphi_i^{\hat{p}-1} - \varphi_{i^*}^{\hat{p}-1}}{\varphi_i^{1/p} - \varphi_{i^*}^{1/p}} \right)^{p-1} \\ &= (1 + (p-1)C_i) \sigma \left( \frac{\varphi_i^{\hat{p}-1} - \varphi_{i^*}^{\hat{p}-1}}{\varphi_i^{\hat{p}-1} - \varphi_{i^*}^{1/p} \varphi_{i^*}^{1/[p(p-1)]}} \right)^{p-1} \\ &\leq (1 + (p-1)C_i) \hat{p}^{p-1} \sigma. \end{aligned}$$

It is clear that  $\varphi^{1/p} \in \mathcal{F}_I$ , we have

$$\lambda_p^{-1} \leq \inf_{f \in \mathcal{F}_I} \sup_{i \in V \setminus \{o\}} I_i(f) \leq \sup_{i \in V \setminus \{o\}} I_i(\varphi^{1/p}) \leq \left( \hat{p}^{p-1} \sup_{i \in V \setminus \{o\}} (1 + (p-1)C_i) \right) \sigma,$$

by Theorem 2.2 (1).

Now, we prove that  $\lambda_p \leq \sigma^{-1}$ . For  $i_0 \in V \setminus \{o\}$ , let  $f$  be a function such that

$$f_i = \begin{cases} \varphi_i^{\hat{p}-1} & \text{if } i \in \mathcal{P}(i_0), \\ \varphi_{i_0}^{\hat{p}-1} & \text{if } i \in V_{i_0}, \\ 0 & \text{Others.} \end{cases}$$

Then

$$\mu(|f|^p) = \sum_{i \in V \setminus \{o\}} \mu_i |f_i|^p = \sum_{i \in \mathcal{P}(i_0)} \mu_i \varphi_i^{\hat{p}} + \mu(T_{i_0}) \varphi_{i_0}^{\hat{p}}.$$

Since  $f_i - f_{i^*} = (v_i)^{\hat{p}-1}$  for  $i \in \mathcal{P}(i_0)$  and  $f_i - f_{i^*} = 0$  for  $i \in V \setminus \mathcal{P}(i_0)$ .

$$D_p(f) = \sum_{i \in V \setminus \{o\}} v_i |f_i - f_{i^*}|^p = \varphi_{i_0}^{\hat{p}-1}.$$

By (3), we have

$$\lambda_p^{-1} \geq \frac{\mu(|f|^p)}{D_p(f)} \geq \mu(T_{i_0}) \varphi_{i_0}, \quad i_0 \in V \setminus \{o\}.$$

Then the assertion follows by taking supremum over  $V \setminus \{o\}$ .  $\square$

## References

- [1] M.F. Chen, *Estimation of spectral gap for Markov chains*, Acta Math. Sin. New Series, 1996, 12:4, 337-360.
- [2] M.F. Chen, *Variational formulas and approximation theorems for the first eigenvalue*, Sci. China (A), 2001, 31:1 (Chinese Edition), 28-36; 44:4 (English Edition), 409-418.
- [3] M.F. Chen, *Speed of stability for birth-death process*, Front. Math. China, 2010, 5:3, 379-516.
- [4] M.F. Chen, L.D. Wang and Y.H. Zhang, *Mixed principle eigenvalues in dimension one*, Front. Math. China, 2013, 8(2): 317-343.
- [5] M.F. Chen, L.D. Wang and Y.H. Zhang, *Mixed eigenvalues of discrete  $p$ -Laplacian*, Front. Math. China, 2014, 9(6): 1261-1292.
- [6] J.H. Shao and Y.H. Mao, *Estimation of the Dirichlet eigenvalue of birth-death process on tree*, Acta mathematical Sinica, Chinese Series. 2007, 50:3, 507-516.
- [7] L.D. Wang and Y.H. Zhang, *The first Dirichlet eigenvalue of birth-death process on trees*, Statistics and Probability Letters. 2013, 83, 1973-1982.